Lecture 24

Consequences of the $\perp$ Decomp Theorem:

Cor: If $W \subset \mathbb{R}^n$ is a subspace then

i) $\dim W + \dim W^\perp = n$

ii) $(W^\perp)^\perp = W$

5.3 Matrix Transposes

Def: If $A = (a_{ij}) = m\times n$ matrix, then the transpose is the $n \times m$ matrix A^T whose ij-entry is a_{ji}.

$\boxed{\text{Ex}}$ $A = \begin{bmatrix} 1 & 0 & 7 \\ -2 & 1 & 4 \end{bmatrix}$, $A_T = \begin{bmatrix} 1 & -2 \\ 0 & 1 \\ 7 & 4 \end{bmatrix}$

i.e. rows of A are cols of $A^\perp$ and vice versa.

Why transposes: $\vec{v}, \vec{w} \in \mathbb{R}^n \rightarrow \vec{v}\cdot\vec{w} = \vec{v}^\perp \vec{w}$

Prop: If $A = m\times n$ matrix then $\forall$ $\vec{v} \in \mathbb{R}^n$ and $\vec{w} \in \mathbb{R}^n$ we have $\vec{v}\cdot(A\vec{w}) = (A^T\vec{v})\cdot\vec{w}$

Proof: Observe

$$\vec{v}\cdot(A\vec{w}) = \vec{v}\,(w_1\vec{a}_1 + \ldots + w_n\vec{a}_n)$$

$$= w_1(\vec{v}\cdot\vec{a}_1) + \ldots + w_n(\vec{v}\cdot\vec{a}_n)$$

$$= \vec{w}\cdot\begin{pmatrix} \vec{v}\cdot\vec{a}_1 \\ \vdots \\ \vec{v}\cdot\vec{a}_n \end{pmatrix} = \vec{w}\cdot(A^T\vec{v}) = (A^T\vec{v})\cdot\vec{w} \qquad \blacksquare$$

Def: A matrix A is **symmetric** if $A^T = A$

$\boxed{\text{e.g.}}$ Identity matrix and zero matrices are symmetric

$\boxed{\text{e.g}}$ $\begin{bmatrix} 1 & 4 & -5 \\ 4 & 2 & 0 \\ -5 & 0 & 3 \end{bmatrix}$ is symmetric

- symmetric implies $n \times n$ (i.e. square)

Properties of transposes: Suppose A and B are (compatibly-sized) matrices and $c \in \mathbb{R}$ is a scaler then:

- $(A^T)^T = A$
- $(cA)^T = c\,A^T$
- $(A+B)^T = A^T + B^T$
- $(AB)^T = B^T A^T$

Proof $(AB)^T = B^T A^T$:

- From proposition, we have $\forall$ (compatibly-sized) vectors $\vec{v}$ and $\vec{w}$ that

On the one hand: $\vec{v} \cdot [(AB)\vec{w}] = [(AB)^T \vec{v}] \cdot \vec{w}$

But on the other hand:

$$\vec{v} \cdot (AB\vec{w})$$
$$= \vec{v} \cdot [A(B\vec{w})]$$
$$= (A^T\vec{v}) \cdot (B\vec{w})$$
$$= (B^T A^T \vec{v}) \cdot \vec{w}$$
$$\rightarrow (AB)^T = B^T A^T$$

<u>Questions</u>: If A, B are $n\times n$ symmetric matrices:

- Is $A+B$ symmetric? yes
- Is AB symmetric? No

<u>Observe</u>: $A = [\vec{a}_1 \cdots \vec{a}_n]$ and then take $A^T A$, then $A^T A$ is symmetric matrix with

ij-entry $= \vec{a}_i \cdot \vec{a}_j$

$\boxed{\text{Ex}}$ $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 1 & 4 \end{bmatrix}$

$$A^T A = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 4 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 1 & 4 \end{bmatrix} = \begin{bmatrix} 2 & 6 \\ 6 & 21 \end{bmatrix}$$

- So if $\vec{a}_1, \dots, \vec{a}_n$ is ortho-normal set of vectors then $A^T A = I$.

<u>Def</u>: $m\times n$ matrix is <u>orthogonal</u> if $A^T A = I$ i.f.f. columns of A form an $\perp$-normal set

$\boxed{\text{Ex}}$

rotation matrices $= \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = \perp$ matrix

Identity in any dimension: $\perp$ matrix

<u>Rmk</u>: If $A = n\times n$ is an $\perp$ matrix then A is invertible and $A^{-1} = A^T$

<u>Questions</u>: Are sums and products of $\perp$ matrices also $\perp$ matrices?

<u>Idea</u>: $n\times n$ $\perp$ matrix is like an n-dimensional rotation. (or reflection)

Cor : If A is an $m \times n$ matrix, then $\forall\ \vec{v}, \vec{w} \in \mathbb{R}^n$ we have :

$$(A\vec{v}) \cdot (A\vec{w}) = \vec{v} \cdot \vec{w}$$

Proof: $(A\vec{v}) \cdot (A\vec{w}) = (A^T A \vec{v}) \cdot \vec{w} = \vec{v} \cdot \vec{w}$ because $A^T A = I$

Cor : If A is an $m \times n$ orthogonal matrix and $\vec{v}, \vec{w} \in \mathbb{R}^n$, then

- $||A\vec{v}|| = ||\vec{v}||$
- $\text{angle}(A\vec{v}, A\vec{w}) = \text{angle}(\vec{v}, \vec{w})$

Prop : Suppose $W \subset \mathbb{R}^n$ is a subspace with ortho-normal basis $\{\vec{u}_1, \dots, \vec{u}_k\}$. Let $Q = [\vec{u}_1 \dots \vec{u}_k]$

Then $\forall\ \vec{v} \in \mathbb{R}^n$ we have :

$$\text{proj}_W(\vec{v}) = Q Q^T \vec{v}$$

i.e. $[\text{proj}_W] = Q Q^T$